

# Critical Point of a Symmetric Vertex Model

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The vertex models have been intensively studied and their thermodynamic properties are well known inside the solvable parameter area.<sup>1,2)</sup> On the other hand, outside the solvable area the model is not fully analyzed. Takasaki et al. numerically investigated a vertex model, which allows 7 vertex configurations and contains 2 parameters.<sup>3)</sup> They observed a phase transition that belongs to the Ising universality class. In this article we study a symmetric vertex model, that allows 10 vertex configurations, by use of the corner transfer matrix renormalization group (CTMRG),<sup>4)</sup> a variant of the density matrix renormalization group.<sup>5-7)</sup> As we report in the following the model has a critical point.

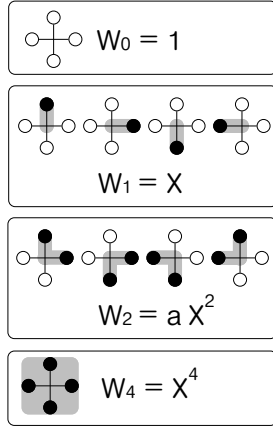


Fig. 1. Allowed vertex configurations and their Boltzmann weights. The open and black circles, respectively, denote spins that take 0 and 1. The shaded area will be used when we plot the spin configuration snapshots in Fig.2 and 5.

Let us introduce the vertex model that we study in the following. Consider a square lattice, where there is a 2-state spin variable ( $\sigma = 0$  or 1) on each bond. Thus a lattice point is surrounded by 4 spins. We impose a local constraint for these 4 spins as shown in Fig.1. We assign the following Boltzmann weights

$$W_0 = 1, \quad W_1 = x, \quad W_2 = ax^2, \quad W_4 = x^4 \quad (1)$$

for these 10 local configurations, where the subscripts denote the sum of 4 spin variables. The parameter  $x$  is positive. The expectation value of a spin

$$\rho(x, a) \equiv \langle \sigma \rangle = (P_1 + 2P_2 + 4P_4)/4, \quad (2)$$

where  $P_\ell$  is the probability to observe vertices with  $\ell$  numbers of 1, is the increasing function of  $x$ .

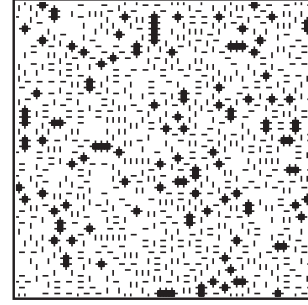


Fig. 2. Spin configuration snapshot at  $a = 0$  and  $x = 1.23$ . For each vertex, the shaded area in Fig.1 is drawn. This configuration snapshot is obtained by recently developed extension of CTMRG.<sup>8,9)</sup>

When the parameter  $a$  is 0, only 6 vertex configurations are allowed. Under this strong constraint, rectangular areas inside which all spins are  $\sigma = 1$ , appear separately in the ‘sea’ of  $\sigma = 0$ , as shown in Fig.2. The expectation value  $\rho(x, a = 0)$  gradually increases with  $x$  till  $x = 1.23$ . At this point the system shows first order phase transition, and  $\rho$  jumps to unity. (See Fig.3.)

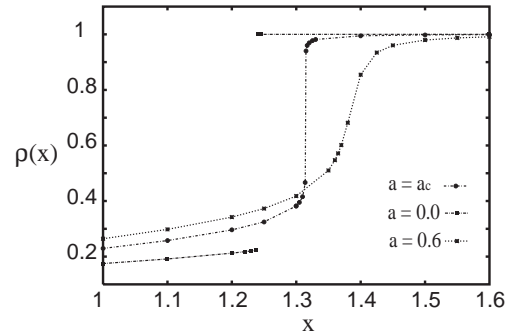


Fig. 3. Expectation value  $\rho(x, a)$  for  $a = 0, 0.3599$ , and 0.6.

In contrast, when  $a = 0.6$  there is no discontinuity nor singularity in  $\rho(x, a = 0.6)$ . Thus there should be a critical point at certain values of  $a$  and  $x$ . We trace the line of the first order transition in the parameter space as shown in Fig.4. The critical value of  $a$  is roughly determined as 0.36. We further perform scaling analysis

for  $\rho(x, a)$ , and find one of the parameter line

$$a = -0.2736(x - x_C) + a_C \quad (3)$$

shown in the inset, that passes the critical point at  $(x_C, a_C) = (1.31438, 0.3599)$ . Figure 5 shows the configuration snapshot at the criticality.

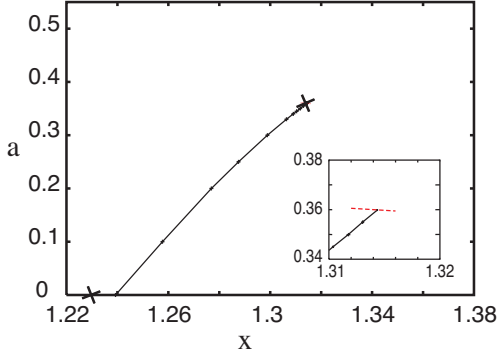


Fig. 4. Phase Diagram. The line of the first order phase transition ends at the critical point  $(x_C, a_C) = (1.31438, 0.3599)$ . Cross marks represent data points used for the snapshot observation in Fig.2 and Fig.5. Inset: the parameter line used for the scaling analysis in Fig.6.

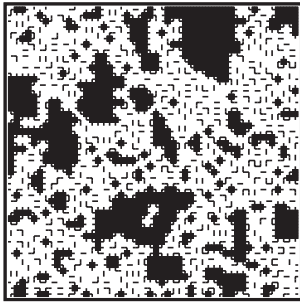


Fig. 5. Snapshot at the criticality.

In order to determine the spin expectation value  $\rho(x_C, a_C)$  at the criticality, we assume a scaling form

$$\rho(x, a) - \rho(x_C, a_C) \propto |x_C - x|^\mu \quad (4)$$

on the parameter line that passes through the critical point,<sup>10)</sup> and plot  $\rho(x, a)$  on the parameter line given by Eq.(3) with respect to a various power of  $|x_C - x|$ . As a result we confirm that  $\rho(x, a)$  shows linear dependence with  $|x_C - x|^{1/15}$  as shown in Fig.6, where the two fitted lines are represented by the following equations

$$\begin{aligned} &0.369(x - x_C)^{1/15} + 0.716 \quad (x > x_C) \\ &-0.422(x_C - x)^{1/15} + 0.716 \quad (x < x_C). \end{aligned} \quad (5)$$

In conclusion the exponent  $\mu$  in Eq.(4) is  $1/15$  and the value  $\rho(x_C, a_C)$  is 0.716. Thus the critical point belongs to the same universality class as the 2D Ising model, whose magnetization at critical temperature obeys the

scaling form

$$M(T_C, h) \propto |h|^{1/\delta} \quad (6)$$

where  $\delta = 15$ .

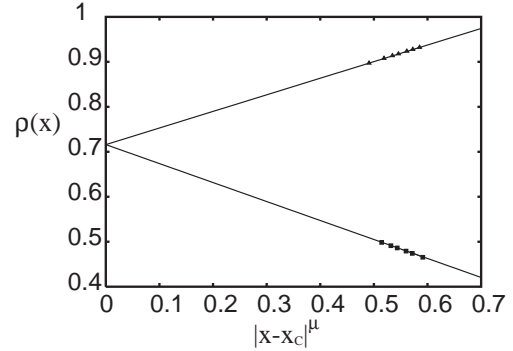


Fig. 6. Spin expectation value  $\rho$  on the parameter line given by Eq.(3). The exponent  $\mu$  is fixed to  $1/15$ .

To summarize, we investigated the phase transition of a vertex model with 10 local configurations, and found a critical point that belongs to the Ising universality class. A possible extension of the current study is toward the direction of the lattice polymer.<sup>11,12)</sup> For example, introducing two new vertices to those vertices shown in Fig.1, one obtains a unified model with the 7-vertex case studied by Takasaki et al. that corresponds to a straight line polymer in two dimension.<sup>3)</sup>

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